Nov 7 2022

Week 10

2020 A Adv. Cal II

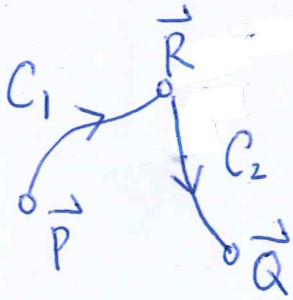
Last time we proved

Theorem 1 Let $\vec{F}$ be a continuous conservative v.f. in open region G. Then

$$\Phi(\vec{Q}) - \Phi(\vec{P}) = \int_C \vec{F}\cdot d\vec{r}, \tag{1}$$

where C is a path from $\vec{Q}$ to $\vec{P}$ in G and Φ is a potential for $\vec{F}$.

But I only showed it when C is a smooth curve. When C is piecewise smooth, say $C = C_1 + C_2$, when C_1, C_2 are smooth, we have



$$\Phi(\vec{R}) - \Phi(\vec{P}) = \int_{C_1} \vec{F}\cdot d\vec{r}$$

$$\Phi(\vec{Q}) - \Phi(\vec{R}) = \int_{C_2} \vec{F}\cdot d\vec{r}$$

Hence

$$\Phi(\vec{Q}) - \Phi(\vec{P}) = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_C \vec{F}\cdot d\vec{r}.$$

Similarly, one can show it in the case $C = C_1 + C_2 + \cdots + C_n$.

A v.f. $\vec{F}$ is independent of path if

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

as long as C_1, C_2 have the same starting and ending points.

<u>Theorem</u> 2. The followings are equivalent.

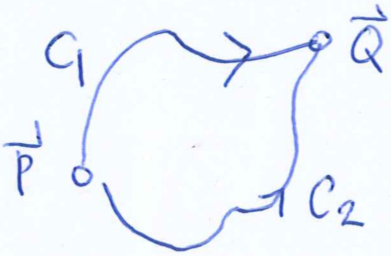
ⓐ $\vec{F}$ is conservative,

ⓑ $\vec{F}$ is independent of path,

ⓒ $\oint_C \vec{F}\cdot d\vec{r} = 0$ for any closed path C.

Pf. ⓐ $\Rightarrow$ ⓑ already done in (1).

ⓑ $\Leftrightarrow$ ⓒ clear.



$C = C_1 + (-C_2)$ is closed.

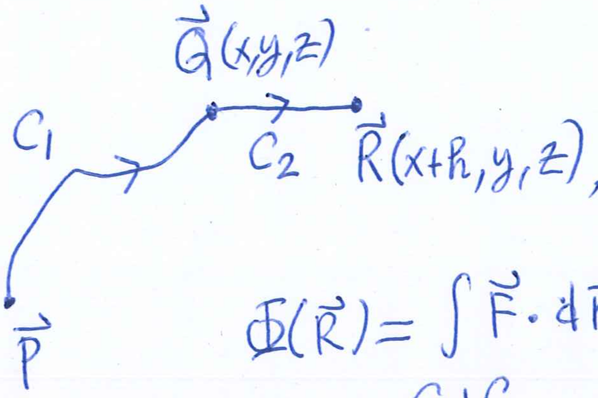
$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1+(-C_2)} \vec{F}\cdot d\vec{r}$$
$$= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r}$$
$$= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}.$$

So $\oint_C \vec{F}\cdot d\vec{r} = 0 \Leftrightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

ⓑ $\Rightarrow$ ⓐ Fix $\vec{P} \in G$. For any $\vec{Q}(x,y,z)$, define

$$\Phi(\vec{Q}) = \int_C \vec{F}\cdot d\vec{r},$$

when C is a path from $\vec{P}$ to $\vec{Q}$ in G. By ⓑ, Φ is well-defined. We verify $\nabla\Phi = \vec{F}$.



$$\Phi(\vec{R}) = \int_{C_1+C_2} \vec{F}\cdot d\vec{r}, \quad \Phi(\vec{Q}) = \int_{C_1} \vec{F}\cdot d\vec{r}$$

$$\begin{aligned}
\therefore\ \Phi(x+h,y,z) - \Phi(x,y,z) &= \Phi(\vec{R}) - \Phi(\vec{Q}) \\
&= \int_{C_1+C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r} \\
&= \int_{C_2} \vec{F}\cdot d\vec{r}.
\end{aligned}$$

C_2 is parametrized by $\vec{r}(t) = (x+th, y, z)$, $t \in [0,1]$. ($\vec{r}(0) = \vec{Q}$, $\vec{r}(1) = \vec{R}$). $\vec{r}'(t) = (h,0,0) = h\hat{i}$.

$$\begin{aligned}
\vec{F}\cdot d\vec{r} &= (M\hat{i} + N\hat{j} + P\hat{k})\cdot h\hat{i} \\
&= Mh.
\end{aligned}$$

$$\therefore\ \Phi(x+h,y,z) - \Phi(x,y,z) = \int_0^1 M(x+th,y,z)\,h\,dt$$

$$\frac{\Phi(x+h,y,z) - \Phi(x,y,z)}{h} = \int_0^1 M(x+th,y,z)\,dt$$

Let $h \to 0$, $\quad \dfrac{\partial \Phi}{\partial x}(x,y,z) = M(x,y,z).$

Similarly, $\quad \dfrac{\partial \Phi}{\partial y} = N, \quad \dfrac{\partial \Phi}{\partial z} = P.$ #

e.g Evaluate

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz .$$

Implicitly it is assumed $y\hat{i} + x\hat{j} + 4\hat{k}$ is conservative, so the line integrals from $(1,1,1)$ to $(2,3,-1)$ are the same.

We find the potential :

$$\frac{\partial \Phi}{\partial x} = y \Rightarrow \Phi = xy + g(y,z)$$

$$\frac{\partial \Phi}{\partial y} = x + \frac{\partial g}{\partial y} = x \Rightarrow \frac{\partial g}{\partial y} = 0, \quad g(y,z) = h(z).$$

$$\frac{\partial \Phi}{\partial z} = 4 \Rightarrow h'(z) = 4 , \quad h(z) = 4z + C$$

$$\therefore \Phi = xy + 4z + C .$$

So,

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = \Phi(2,3,-1) - \Phi(1,1,1)$$

$$= (2\times 3 + 4\times -1 + C) - (1 + 4 + C)$$

$$= -3 \ \#$$

Note :

A formal expression

$$M\,dx + N\,dy + P\,dz$$

is called a <u>differential form</u>. A differential form is <u>exact</u> if $M\hat{i} + N\hat{j} + P\hat{k}$ is conservative, and we write

$$d\Phi = M\,dx + N\,dy + P\,dz .$$

Last time we showed that for a smooth v.f. on G, if it is conservative, then the component test holds:

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},\ \frac{\partial M}{\partial z}=\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial z}=\frac{\partial P}{\partial y}.$$

(Indeed, let $\vec{F}=(F_1,\cdots,F_n)\in\mathbb{R}^n$, this condition becomes

$$\frac{\partial F_i}{\partial x_j}=\frac{\partial F_j}{\partial x_i},\ i\neq j.\ )$$

We note

- There are cases the component test holds but the v.f. is not conservative (see below)
- When the v.f. is defined in the entire space, then v.f is conservative iff it passes the component test (see Ex 9).

<u>Example</u> Consider $\vec{F}=\frac{-y}{x^2+y^2}\hat{i}+\frac{x}{x^2+y^2}\hat{j}+0\hat{k}$.

$\vec{F}$ is NOT defined in $\mathbb{R}^3$ but in $\mathbb{R}^3$ omitting the z-axis.

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\frac{-y}{x^2+y^2}=\frac{-1}{x^2+y^2}-\frac{-y(2y)}{(x^2+y^2)^2}=\frac{-x^2+y^2}{(x^2+y^2)^2},$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\frac{x}{x^2+y^2}=\frac{1}{x^2+y^2}-\frac{x(2x)}{(x^2+y^2)^2}=\frac{-x^2+y^2}{(x^2+y^2)^2}.$$

$\therefore\ M_y=N_x$, and $M_z=0=P_x$, $N_z=0=P_y$.

So the component test holds.

But, claim it is not conservative.

Consider the closed path $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+0\hat{k}$,

$t\in[0,2\pi]$.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t+\sin^2 t}\hat{\imath} + \frac{\cos t}{\cos^2 t+\sin^2 t}\hat{\jmath} \right)\cdot\left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}\right) dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= 2\pi \neq 0.$$

By thm 2, $\vec{F}$ is not conservative.

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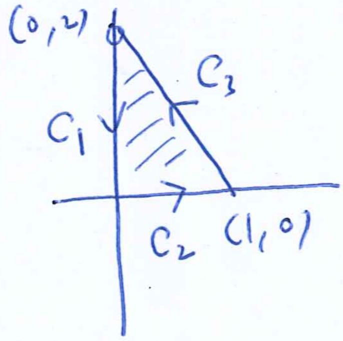
Green's theorem relates line integral to double integral.

Theorem (Green's theorem) Let $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ be a smooth v.f. defined on the simple closed curve C as well as the region D it bounds. Then

$$\oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\, dA(x,y),$$

where C is in anticlockwise way.

e.g. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = x\hat{\imath} + y\hat{\jmath}$, $C = C_1 + C_2 + C_3$.



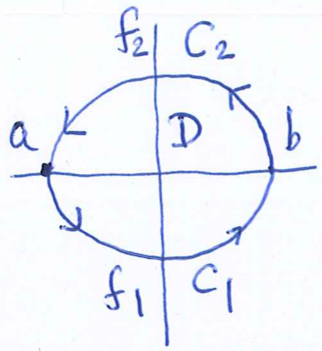
To avoid doing line integral three times, we apply Green's thm

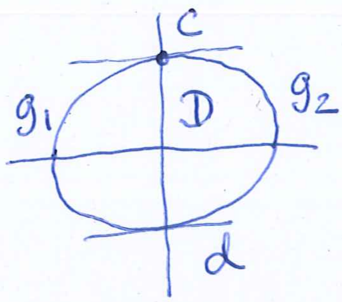
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) dA = 0.$$

Pf. We verify Green's thm when D can be described by

$$D = \{(x,y) : f_1(x) \le y \le f_2(x),\ a \le x \le b\}, \quad f_1(a)=f_2(a),\ f_1(b)=f_2(b)$$

$$= \{(x,y) : g_1(y) \le x \le g_2(y),\ c \le y \le d\} \quad g_1(c)=g_2(c),\ g_1(d)=g_2(d).$$





Focusing on the 1st figure, claim

$$-\iint_D M_y\, dA = \oint_C M\, dx \qquad (2)$$

Focusing on the 2nd figure, claim

$$\iint_D N_x\, dA = \oint_C N\, dy \qquad (3)$$

(2)+(3) $\Rightarrow$ Green's theorem.

Pf of (2):

$$\iint_D M_y\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx$$

$$= \int_a^b M(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx$$

On the other hand,

$C = C_1 + C_2$, $\quad C_1 \ \vec{r}_1(x) = (x, f_1(x)),\ x \in [a,b],\ \vec{r}_1'(x) = (1, f_1'(x))$

$\quad -C_2 \ \vec{r}_2(x) = (x, f_2(x)),\ x \in [a,b].\ \vec{r}_2'(x) = (1, f_2'(x))$

$$\therefore \oint_C M\,dx = \int_{C_1} M\,dx + \int_{C_2} M\,dx$$

$$= \int_{C_1} M\,dx - \int_{-C_2} M\,dx$$

$$= \int_a^b M(x, f_1(x)) \cdot 1\,dx - \int_a^b M(x, f_2(x)) \cdot 1\,dx$$

$$\therefore \iint_D M_y\,dA = -\oint_C M\,dx, \quad (2) \text{ holds.}$$

Similarly, one shows (3) . #

(Con'td)

Consequences of Green's theorem.

(I) A region is called <u>simply-connected</u> if any closed loop in it can be continuously deformed to a single point in the region. Geometrically it means the region has no holes nor punctured.

$\mathbb{R}^2$ removed the negative x-axis.

Above are simply-connected region.

1 holes

punctured one point

2 holes

$\mathbb{R}^2$ removed $(0,0)$.

non simply-connected region

Theorem Let $\vec{F}$ be a smooth v.f. in a simply-connected region D. Then $M_y = N_x$ implies $\vec{F}$ is conservative.

Pf. Let C be a simple closed curve in D. It suffices to show

$$\oint_C \vec{F}\cdot d\vec{r} = 0.$$

However, the region D_1 enclosed by C is contained in D (since D has no holes/punctured). By Green's thm.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (N_x - M_y)\,dA = 0. \quad \#$$

Note. The v.f. $\dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$

is defined in $\mathbb{R}^2$ with the origin removed (not simply connected).

If we consider the smaller region

$$D = \{(x,y) : (x,y) \neq (x,0),\ x \leq 0\}, \text{ that is, } \mathbb{R}^2$$

with the negative x-axis removed. D is simply-connected.

You are asked in the exercise to find a potential in D.

II Localization of the circulation.

The circulation of a closed curve is defined to be

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy$$

$$= \oint_C \vec{F}\cdot\hat{t}\,ds$$

To define the "circulation" of $\vec{F}$ at (x,y) we enclose (x,y) by a closed curve C and shrink C to (x,y)

$$\frac{1}{|D|}\oint \vec{F}\cdot d\vec{r} = \frac{1}{|D|}\iint_D (N_x - M_y)\,dA$$

$$\rightarrow (N_x - M_y)(x,y) \quad \text{as } |D| \rightarrow 0$$

It suggests to define the circulation / the <u>curl</u> of $\vec{F}$ at (x,y) to be

$$\text{curl}\,\vec{F} = (N_x - M_y)(x,y).$$

Then $\iint_D \text{curl}\,\vec{F} = \oint_C \vec{F}\cdot d\vec{r}$ gives the circulation.

Another form of Green's theorem.
Letting $M \rightarrow -N$, $N \rightarrow M$, Green's then becomes

$$\iint_D M_x - (-N)_y = \oint_C -N\,dx + M\,dy,$$

the RHS is the outward flux of $\vec{F}$ across C. So after localization, the <u>divergence</u> of $\vec{F}$ at (x,y) is

$$\text{div}\,\vec{F}(x,y) \equiv M_x + N_y.$$

So, $\iint_D \text{div}\,\vec{F}\,dA = \oint_C -N\,dx + M\,dy$

$= \oint_C \vec{F}\cdot\hat{n}\,ds$, the outward flux of $\vec{F}$ across C.

e.g. Find the outward flux of $2e^{xy}\hat{i}+y^3\hat{j}$ across the square $x=\pm 1$, $y=\pm 1$.

$M = 2e^{xy}$, $N = y^3$.

$\operatorname{div}\vec{F} = M_x + N_y = 2ye^{xy} + 3y^2$

$\therefore$ flux $= \int_{-1}^{1}\int_{-1}^{1} (2ye^{xy} + 3y^2)\,dx\,dy$

$\vdots$

$= 4.$ #